

# On a real multiplication problem

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## Abstract

An explicit class field theory for the real quadratic number fields is developed. The construction is based on the theory of the Hecke eigenforms (of weight two) and the notion of a pseudo-lattice with the real multiplication introduced by Yu. I. Manin. In particular, it is shown how to extend the domain of definition of the  $j$ -invariant to the quadratic irrational points at the boundary of the upper half-plane  $\mathbb{H}$ .

*Key words and phrases:* pseudo-lattices, measured foliations

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## Introduction

**A. The explicit class field theory.** Given the algebraic number field  $k$ , how to construct an abelian extension of  $k$ , i.e. the extension,  $K$ , whose Galois group is an abelian group? For any ground field  $k$ , a general answer is given by the class field theory of  $k$ . However, an explicit construction of the generators of  $K$  is an open problem, except for the case  $k = \mathbb{Q}$  and  $k = \mathbb{Q}(\sqrt{-d})$  (an imaginary quadratic number field). Namely, the Kronecker-Weber theorem says that every finite abelian extension of the field of rational numbers  $\mathbb{Q}$  is a subfield of the field  $\mathbb{Q}(\zeta_n)$ , where  $\zeta_n$  is a primitive root of unity of the order  $n$ . It is well-known that the abelian extensions of the imaginary quadratic number fields can be obtained using the theory of elliptic curves with complex multiplication. The case of the abelian extensions of real quadratic number fields has been studied by Shimura [11]. It was noticed, that such extensions can be generated by the coordinates of certain points of finite order on an abelian variety associated to the Weil differential  $\omega_N$  on a modular curve  $X_0(N)$ . Around 1970s, Stark has formulated a series of conjectures on the abelian extensions of arbitrary number fields. In particular, the abelian extensions over the real quadratic number fields were conjectured to arise from the special values of the Artin  $L$ -functions attached to the real quadratic number field  $k$ . These special values coincide with the units of a number field (Stark's units) and generate the abelian extension [12].

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**B. The Manin pseudo-lattices.** Recall that a pseudo-lattice can be defined as a  $\mathbb{Z}$ -module  $\mathfrak{m} = \mathbb{Z} + \mathbb{Z}\theta$  of the real line  $\mathbb{R}$  taken up to a scaling. It is verified directly, that the ring of endomorphisms of  $\mathfrak{m}$  is bigger than the ring  $\mathbb{Z}$ , if and only if  $\theta$  is a quadratic irrationality. In this case the pseudo-lattice is said to have a *real multiplication*. Around 2002, Yu. I. Manin [8] suggested to use the pseudo-lattices with a real multiplication as a replacement for the lattices with the complex multiplication. (The complex multiplication solves explicitly the class field theory for the imaginary quadratic fields.) The idea is that the real multiplication will play a decisive rôle in the construction of the generators of the abelian extensions of the real quadratic fields. For brevity, we call any pseudo-lattice with the real multiplication a *Manin pseudo-lattice* [8].

**C. The real multiplication problem.** Let  $\mathfrak{m}$  be a Manin pseudo-lattice and  $R = \text{End}(\mathfrak{m})$  its endomorphism ring. Denote by  $k = R \otimes_{\mathbb{Z}} \mathbb{Q}$  the real quadratic number field associated to the ring  $R$ . Note that  $R = \mathbb{Z} + \mathfrak{f}O_k$  is an order in  $k$ , where  $\mathfrak{f} \geq 1$  is the conductor of the order. The real multiplication problem can be formulated as follows.

**Main problem.** *To construct an abelian extension (a ring class field modulo  $\mathfrak{f}$ ) of the field  $k$ .*

The aim of the article is a solution to the problem. Our approach is based on geometry of the Hecke eigenforms on a modular curve  $X_0(N)$ . An outline of the main ideas is given below. (An exact formulation can be found in section 1.)

**D. The Anosov-Hecke eigenforms.** Let  $N > 0$  be an integer and  $\Gamma_0(N)$  the Hecke subgroup of the modular group  $SL_2(\mathbb{Z})$ . By  $S_2(\Gamma_0(N))$  one understands a collection of the cusp forms (of the weight two) on the extended upper half-plane  $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ . Recall that the Riemann surface  $X_0(N) := \mathbb{H}^*/\Gamma_0(N)$  is called a *modular curve*. We identify the  $S_2(\Gamma_0(N))$  and the linear space  $\Omega_{hol}(X_0(N))$  of the holomorphic differentials on  $X_0(N)$  via the formula  $f(z) \mapsto \omega = f(z)dz$ . By  $\mathbb{T}_{\mathbb{Z}} := \mathbb{Z}[T_1, T_2, \dots]$  one denotes the commutative algebra generated by the Hecke operators,  $T_n$ , acting on the space  $S_2(\Gamma_0(N))$ . The common eigenvector  $f \in S_2(\Gamma_0(N))$  of all  $T_n \in \mathbb{T}_{\mathbb{Z}}$  is referred to as a *Hecke eigenform*. By  $K_f$  we understand an algebraic number field generated by the Fourier coefficients of the Hecke eigenform  $f$ . Let  $g > 1$  be the genus of  $X_0(N)$ . It is well known that  $\deg(K_f|\mathbb{Q}) \leq g$ , see e.g. [2], p. 25. The eigenform will be called an *Anosov-Hecke eigenform*, if  $\deg(K_f|\mathbb{Q}) = g$ . (The prefix ‘Anosov’ hinges on the fact, that  $\text{Re } \omega = 0$  gives an Anosov foliation on  $X_0(N)$  [9].)

**E. The Hecke units.** Let  $\omega = f(z)dz$  be a holomorphic differential on  $X_0(N)$ , corresponding to the Anosov-Hecke eigenform  $f$ . Consider a  $\mathbb{Z}$ -module  $\int \text{Re } \omega = \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_g$ , where the integration is taken over the (relative) homology group  $H_1(X_0(N), \text{Sing } \omega; \mathbb{Z})$ . The  $\mathbb{Z}$ -module lies in the field  $K_f$  (lemma 3) and the vector  $\lambda = (\lambda_1, \dots, \lambda_g)$  can be interpreted as the Perron-Frobenius eigenvector of an integer matrix  $A$ . The eigenvalue  $\lambda_A > 1$ , which satisfies the equation  $A\lambda = \lambda_A \lambda$ , will be called a *Hecke unit* of the eigenform  $f$ . The Hecke unit does not depend on the choice of basis in the  $\mathbb{Z}$ -module  $\mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_g$ . The

$\lambda_A \in K_f$  is a unit of the endomorphism ring of the  $\mathbb{Z}$ -module  $\mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_g$ , hence the name.

**F. The Hecke pseudo-lattices.** Let  $T^2$  be a two-torus. Consider a continuous map  $h : X_0(N) \rightarrow T^2$  ramified at a single point of  $T^2$  (see section 1 for the details). Note that the closed form  $Re \omega$  on the  $X_0(N)$  covers a closed form  $\phi = h(Re \omega)$  on  $T^2$ . The pseudo-lattice  $\mathfrak{m}_H := \int \phi = \mathbb{Z} + \mathbb{Z}\theta$  will be called a *Hecke pseudo-lattice*. The integration is taken over  $H_1(T^2; \mathbb{Z})$  and  $\phi$  is normalized, so that 1 is a generator of the  $\mathbb{Z}$ -module. The basic lemma says, that the  $\mathfrak{m}_H$  is a Manin pseudo-lattice (lemma 6). Finally, a *j-invariant* of  $\mathfrak{m}_H$  is defined as  $j(\mathfrak{m}_H) := \lambda_A$ , where  $\lambda_A$  is the Hecke unit. The *j-invariant* is independent of the choice of the generator  $\theta$  in  $\mathfrak{m}_H$ . In this way, one gets a natural extension of the domain of classical *j-invariant* to the quadratic irrational points at the boundary of  $\mathbb{H}$ .

**G. The ring class field of a real quadratic field.** Let  $\mathfrak{m}_H$  be a Hecke pseudo-lattice and  $End(\mathfrak{m}_H) = R$ . Consider the real quadratic number field  $k = R \otimes_{\mathbb{Z}} \mathbb{Q}$  and let  $\mathfrak{f} \geq 1$  be the conductor  $R$  in the field  $k$ . Our main result is contained in the following theorem.

**Theorem 1** *The number field  $K = k(j(\mathfrak{m}_H))$  is a ring class field (modulo  $\mathfrak{f}$ ) of the real quadratic field  $k$ .*

**H. The structure of the paper.** The structure of the article is as follows. In section 1, the preliminary results are formulated. Theorem 1 is proved in section 2.

## 1 Notation

### 1.1 The Anosov-Hecke eigenforms

**A. Foliations.** By a  $p$ -dimensional, class  $C^r$  foliation of an  $m$ -dimensional manifold  $M$  one understands a decomposition of  $M$  into a union of disjoint connected subsets  $\{\mathcal{L}_\alpha\}_{\alpha \in A}$ , called the *leaves* of the foliation. The leaves must satisfy the following property: Every point in  $M$  has a neighborhood  $U$  and a system of local class  $C^r$  coordinates  $x = (x^1, \dots, x^m) : U \rightarrow \mathbb{R}^m$  such that for each leaf  $\mathcal{L}_\alpha$ , the components of  $U \cap \mathcal{L}_\alpha$  are described by the equations  $x^{p+1} = Const, \dots, x^m = Const$ . Such a foliation is denoted by  $\mathcal{F} = \{\mathcal{L}_\alpha\}_{\alpha \in A}$ . The number  $q = m - p$  is called a *codimension* of the foliation  $\mathcal{F}$ , see [5] p.370. The codimension  $q$   $C^r$ -foliations  $\mathcal{F}_0$  and  $\mathcal{F}_1$  are said to be  $C^s$ -conjugate ( $0 \leq s \leq r$ ) if there exists a diffeomorphism of  $M$ , of class  $C^s$ , which maps the leaves of  $\mathcal{F}_0$  onto the leaves of  $\mathcal{F}_1$ . If  $s = 0$ ,  $\mathcal{F}_0$  and  $\mathcal{F}_1$  are *topologically conjugate*, *ibid.*, p.388.

**B. The singular foliations.** The foliation  $\mathcal{F}$  is called *singular* if the codimension  $q$  of the foliation depends on the leaf. We further assume that  $q$  is constant for all but a *finite* number of leaves. Such a set of the exceptional leaves will be

denoted by  $Sing \mathcal{F} := \{\mathcal{L}_\alpha\}_{\alpha \in F}$ , where  $|F| < \infty$ . Note that in the case  $F$  is an empty set, one gets the usual definition of a (non-singular) foliation. A quick example of the singular foliations is given by the trajectories of a non-trivial differential form on the manifold  $M$ , which vanish in a finite number of the points of  $M$ . It is easy to see that the set of zeroes of such a form corresponds to the exceptional leaves of the foliation.

**C. The measured foliations.** Roughly, the measured foliation is a singular codimension 1  $C^r$ -foliation, induced by the trajectories of a closed differential form  $\phi$  on a two-dimensional manifold (surface),  $X$ . Namely, a *measured foliation*,  $\mathcal{F}$ , on a surface  $X$  is a partition of  $X$  into the singular points  $x_1, \dots, x_n$  of order  $k_1, \dots, k_n$  and regular leaves (1-dimensional submanifolds). On each open cover  $U_i$  of  $X - \{x_1, \dots, x_n\}$  there exists a non-vanishing real-valued closed 1-form  $\phi_i$  such that: (i)  $\phi_i = \pm \phi_j$  on  $U_i \cap U_j$ ; (ii) at each  $x_i$  there exists a local chart  $(u, v) : V \rightarrow \mathbb{R}^2$  such that for  $z = u + iv$ , it holds  $\phi_i = Im(z^{\frac{k_i}{2}} dz)$  on  $V \cap U_i$  for some branch of  $z^{\frac{k_i}{2}}$ . The pair  $(U_i, \phi_i)$  is called an atlas for the measured foliation  $\mathcal{F}$ . Finally, a measure  $\mu$  is assigned to each segment  $(t_0, t) \in U_i$ , which is transverse to the leaves of  $\mathcal{F}$ , via the integral  $\mu(t_0, t) = \int_{t_0}^t \phi_i$ . The measure is invariant along the leaves of  $\mathcal{F}$ , hence the name. Note that the measured foliation can have singular points with an odd number of the saddle sections. (Those cannot be continuously oriented along the leaves and therefore cannot be given by the trajectories of a closed form.) However, when all  $k_i$  are even integers, the measured foliation can be prescribed a continuous orientation and is called *oriented*. Such foliations are given by the trajectories of a closed differential form on the surface  $X$ . In what follows, we shall work with the class of oriented measured foliations.

**D. The jacobian of a measured foliation.** Let  $\mathcal{F}$  be a measured foliation on a compact surface  $X$ . We shall assume that  $\mathcal{F}$  is an oriented foliation, i.e. given by the trajectories of a closed form  $\phi$  on  $X$ . The assumption is no restriction – each measured foliation is oriented on a surface  $\tilde{X}$ , which is a double cover of  $X$  ramified at the singular points of the half-integer index of the non-oriented foliation [4]. Let  $\{\gamma_1, \dots, \gamma_n\}$  be a basis in the relative homology group  $H_1(X, Sing \mathcal{F}; \mathbb{Z})$ , where  $Sing \mathcal{F}$  is the set of singular points of the foliation  $\mathcal{F}$ . It is well known (*ibid.*), that  $n = 2g + m - 1$ , where  $g$  is the genus of  $X$  and  $m = |Sing(\mathcal{F})|$ . The periods of  $\phi$  in the above basis we shall write as:  $\lambda_i = \int_{\gamma_i} \phi$ . It is known that the reals  $\lambda_i$  are coordinates of the foliation  $\mathcal{F}$  in the space of all measured foliations on the surface  $X$  (with a given set of the singular points) [3]. By a *jacobian*,  $Jac(\mathcal{F})$ , of the measured foliation  $\mathcal{F}$ , we understand a  $\mathbb{Z}$ -module  $\mathfrak{m} = \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_n$  regarded as a subset of the real line  $\mathbb{R}$ .

**E. Properties of the jacobian.** An importance of the jacobians stems from the fact that although the periods  $\lambda_i$  depend on the basis in  $H_1(X, Sing \mathcal{F}; \mathbb{Z})$ , the jacobian does not. Moreover, up to a scalar multiple, the jacobian is an invariant of the topological conjugacy of the foliation  $\mathcal{F}$ . We shall formalize the observations in the following two lemmas.

**Lemma 1 ([9])** *The  $\mathbb{Z}$ -module  $\mathfrak{m}$  is independent of the choice of a basis in the homology group  $H_1(X, \text{Sing } \mathcal{F}; \mathbb{Z})$  and depends solely on the foliation  $\mathcal{F}$ .*

Recall that the measured foliations  $\mathcal{F}$  and  $\mathcal{F}'$  are topologically conjugate, if there exists an automorphism  $h$  of the surface  $X$ , which sends the leaves of the foliation  $\mathcal{F}$  to the leaves of the foliation  $\mathcal{F}'$ . Note that the conjugacy deals with the topological foliations (i.e. the projective classes of the measured foliations) and does not preserve the transversal measure of the leaves.

**Lemma 2 ([9])** *The measured foliations  $\mathcal{F}, \mathcal{F}'$  on the surface  $X$  are topologically conjugate if and only if  $\text{Jac } (\mathcal{F}') = \mu \text{ Jac } (\mathcal{F})$ , where  $\mu > 0$  is a real number.*

**F. Foliations given by the cusp forms.** It follows from the Hubbard-Masur main theorem [4], that the space  $S_2(\Gamma_0(N))$  is isomorphic to the space  $\Phi_{X_0(N)}$  of all measured foliations generated by the real part of the cusp forms. In particular,  $\dim_{\mathbb{R}}(S_2(\Gamma_0(N))) = \dim(\Phi_{X_0(N)}) = 2g$ , where  $g$  is the genus of  $X_0(N)$ . But  $\dim(\Phi_{X_0(N)}) = \text{rank}(\text{Jac } (\mathcal{F})) = 2g + m - 1$ . Thus, we conclude that  $m = 1$ , where  $m$  is the number of the singular points of the foliation  $\mathcal{F}$ . (The reader must not be confused comparing this result with the fact that the zeroes of the holomorphic form  $\omega = f(z)dz$  are bijective with the cusps of  $X_0(N)$ , whose number can exceed one. In the case of more than one cusp, the measured foliation with the singular saddle points, situated in the cusps, is measure equivalent to the foliation with the unique singular point via a homotopy along the saddle connections between the cusps.)

Recall that there exists a natural involution  $i$  on the space  $S_2(\Gamma_0(N))$  defined by the formula  $f(z) \mapsto f^*(z)$ , where  $f(z) = \sum c_n q^n$  and  $f^*(z) = \sum \bar{c}_n q^n$ . A subspace,  $S_2^{\mathbb{R}}(\Gamma_0(N))$ , fixed by the involution, consists of the cusp forms, whose Fourier coefficients are the real numbers. Clearly,  $\dim_{\mathbb{R}}(S_2^{\mathbb{R}}(\Gamma_0(N))) = g$ . The  $i$  induces an involution,  $i_{\Phi}$ , on the space  $\Phi_{X_0(N)}$ , which (in a proper system of the coordinates) acts by the formula  $(\lambda_1, \dots, \lambda_g, \lambda'_1, \dots, \lambda'_g) \mapsto (\lambda'_1, \dots, \lambda'_g, \lambda_1, \dots, \lambda_g)$ . It is easy to see, that the  $i_{\Phi}$ -invariant subspace,  $\Phi_{X_0(N)}^{\mathbb{R}}$ , consists of the measured foliations  $(\lambda_1, \dots, \lambda_g, \lambda_1, \dots, \lambda_g)$ . Thus,  $\text{Jac } (\mathcal{F}) = \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_g$  for  $\forall \mathcal{F} \in \Phi_{X_0(N)}^{\mathbb{R}}$ .

**G. The jacobian of the foliation given by the Hecke eigenform.** Let  $f \in S_2(\Gamma_0(N))$  be a (normalized) Hecke eigenform, such that  $f(z) = \sum_{n=1}^{\infty} c_n(f)q^n$  its Fourier series. We shall denote by  $K_f = \mathbb{Q}(\{c_n(f)\})$  the algebraic number field generated by the Fourier coefficients of  $f$ . Let  $g$  be the genus of the modular curve  $X_0(N)$ . It is well known that  $1 \leq \deg(K_f | \mathbb{Q}) \leq g$  and  $K_f$  is a totally real field, see e.g. [2], p. 25. Let  $\mathcal{F}$  be the foliation given by the lines  $\text{Re}(f dz) = 0$ . The following lemma will be critical.

**Lemma 3 ([7], Theorem 1.2), ([9])**  *$\text{Jac } (\mathcal{F})$  is a  $\mathbb{Z}$ -module in the field  $K_f$ .*

## 1.2 Hecke units

**A. The Jacobi-Perron continued fractions ([1], [10]).** Let  $\lambda = (\lambda_1, \dots, \lambda_n)$ ,

$\lambda_i \in \mathbb{R} - \mathbb{Q}$  and  $\theta_{i-1} = \frac{\lambda_i}{\lambda_1}$ , where  $\lambda_i \geq 0$ ,  $\lambda_1 \neq 0$  and  $1 \leq i \leq n$ . The continued fraction

$$\begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_{n-1} \end{pmatrix} = \lim_{k \rightarrow \infty} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & b_1^{(1)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b_{n-1}^{(1)} \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & b_1^{(k)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b_{n-1}^{(k)} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

where  $b_i^{(j)} \in \mathbb{N} \cup \{0\}$ , is called the *Jacobi-Perron fraction*. To recover the integers  $b_i^{(k)}$  from the vector  $(\theta_1, \dots, \theta_{n-1})$ , one has to repeatedly solve the following system of equations:  $\theta_1 = b_1^{(1)} + \frac{1}{\theta'_1}, \theta_2 = b_2^{(1)} + \frac{\theta'_1}{\theta'_{n-1}}, \dots, \theta_{n-1} = b_{n-1}^{(1)} + \frac{\theta'_{n-2}}{\theta'_{n-1}}$ , where  $(\theta'_1, \dots, \theta'_{n-1})$  is the next input vector. Thus, each vector  $(\theta_1, \dots, \theta_{n-1})$  gives rise to a formal Jacobi-Perron continued fraction. Let  $B_k = \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix}$ . If there exists the number  $n > 0$  such that  $B_k = B_{k+p}$  for all  $k$  except a finite number, then the Jacobi-Perron fraction is called *periodic*. Let  $A = B_k \dots B_{k+p}$ , where  $p$  is the minimal period of a periodic Jacobi-Perron continued fraction. The equation  $\det(A - \lambda I) = 0$  defines a polynomial in  $\lambda$ , which is called a characteristic polynomial of the periodic Jacobi-Perron fraction. Let  $\lambda_A > 1$  be the Perron-Frobenius root of the characteristic polynomial of the periodic Jacobi-Perron continued fraction and consider the corresponding eigenvalue problem  $A - \lambda_A I = 0$ . Then the periodic continued fraction converges to the Perron-Frobenius eigenvector  $(\alpha_1, \dots, \alpha_n)$ , where  $\alpha_i \in \mathbb{Q}(\lambda_A)$ , see [10], Satz XII.

**B. The Hecke unit of the Anosov-Hecke eigenform.** Let  $f$  be an Anosov-Hecke eigenform. Recall that  $Jac(\mathcal{F}) := \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_g \subset K_f$ , where  $\mathcal{F}$  is the measured foliation given by the real part of the holomorphic differential  $\omega = f dz$  and  $K_f$  is an algebraic number field (of degree  $g$ ) generated by the Fourier coefficients of  $f$ . One can always assume that  $\lambda_i > 0$ . (For otherwise a change of the basis in the  $\mathbb{Z}$ -module is required.) The vector  $\lambda = (\lambda_i)$  unfolds into a periodic Jacobi-Perron continued fraction with the period matrix  $A$ . It is immediate that  $\lambda$  is the Perron-Frobenius eigenvector of  $A$ , i.e.  $A\lambda = \lambda_A \lambda$  for a  $\lambda_A > 1$ . The eigenvalue  $\lambda_A \in K_f$  we shall call a *Hecke unit* of  $f$ .

**Lemma 4** *The Hecke unit is an invariant of the  $\mathbb{Z}$ -module  $Jac(\mathcal{F})$ .*

*Proof.* Let  $\lambda' = B\lambda$  be a new basis of the  $\mathbb{Z}$ -module  $Jac(\mathcal{F})$ . It is easy to see, that  $A' = BAB^{-1}$  and  $A$  are similar matrices. In particular,  $A'\lambda' = \lambda_A \lambda'$ . Thus,  $\lambda_A$  is an invariant of the  $\mathbb{Z}$ -module  $Jac(\mathcal{F})$ .  $\square$

### 1.3 Hecke pseudo-lattices

**A. The ramified cover of the torus.** Recall that an  $n$ -fold continuous map  $h : X \rightarrow Y$  is called ramified at the points  $y_1, \dots, y_n \in Y$ , if  $h$  is  $n$ -to-one everywhere except  $y_i$ , where it is one-to-one. Let  $X = X_0(N)$ ,  $Y = T^2$

(a two-dimensional torus) and  $y_i = y$  be a single point. It follows from the Riemann-Hurwitz formula, that in this case  $n = 2g - 1$ , where  $g$  is the genus of the surface  $X_0(N)$ . The map  $h$  is unique modulo a homotopy. Let  $\mathcal{F}$  be the foliation on the  $X_0(N)$ , given by a cusp form  $f \in S_2(\Gamma_0(N))$ . Then, as explained in §1.1.F, the set  $Sing \mathcal{F}$  consists of the unique multi-saddle. The saddle point can be taken for the ramification point of the map  $h$ . Under the map  $h$ , The multi-saddle covers a two-saddle (a fake saddle) on the  $T^2$  (Fig. 1).

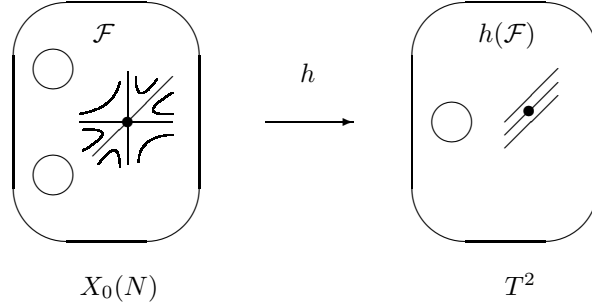


Figure 1: The map  $h$  (case  $g = 2$ ).

**B. A pseudo-lattice given by the map  $h$ .** The map  $h$  defines a foliation,  $h(\mathcal{F})$  on  $T^2$ , which is covered by the foliation  $\mathcal{F}$ . Since  $\mathcal{F}$  is a measured foliation, such will be the  $h(\mathcal{F})$ . The  $Jac(h(\mathcal{F})) := \mathbb{Z} + \mathbb{Z}\theta$  is a pseudo-lattice, where  $\theta$  has a geometric meaning of the slope of the leaves of  $h(\mathcal{F})$  on the torus.

**Lemma 5**  *$Jac(h(\mathcal{F}))$  does not depend on the homotopy class of the map  $h$ .*

*Proof.* Note that any automorphism  $g \in Mod(X_0(N))$  covers an automorphism  $h(g) \in Mod(T^2)$ . The foliations  $h(\mathcal{F})$  and  $h(g(\mathcal{F}))$  are therefore topologically conjugate. By lemma 2, their jacobians are proportional.  $\square$

**C. The basic lemma.** Given an Anosov-Hecke eigenform  $f \in S_2(\Gamma_0(N))$ , we shall call  $\mathfrak{m}_H := \mathbb{Z} + \mathbb{Z}\theta$  obtained from  $f$  a *Hecke pseudo-lattice*. The following lemma establishes the main property of the Hecke pseudo-lattices.

**Lemma 6** *The  $\mathfrak{m}_H$  is a Manin pseudo-lattice.*

*Proof.* (i) First, let us show that  $Jac(h(\mathcal{F})) \subseteq Jac(\mathcal{F})$  is an inclusion of the  $\mathbb{Z}$ -modules. Indeed, let  $h_* : H_1(X_0(N), Sing \mathcal{F}; \mathbb{Z}) \rightarrow H_1(T^2, Sing h(\mathcal{F}); \mathbb{Z})$  be the action of the cover map  $h$  on the first homology of the respective surfaces.

Suppose that  $\gamma \in H_1(X_0(N), \text{Sing } \mathcal{F}; \mathbb{Z})$  is such that  $h_*(\gamma) \neq 0$ , i.e.  $\gamma$  is not in the kernel of the linear map  $h_*$ . One can normalize  $h$ , so that  $\int_\gamma f dz = \int_{h_*(\gamma)} h(f dz)$ , see e.g. [6], p.19. Taking the real parts of the above integrals, one gets  $\int_\gamma \text{Re}(f dz) = \int_{h_*(\gamma)} \text{Re}(h(f dz)) := \theta$ . Thus,  $\theta$  is a generator in the  $\mathbb{Z}$ -modules  $Jac(\mathcal{F})$  and  $Jac(h(\mathcal{F}))$ . On the other hand, the  $\mathbb{Z}$ -modules  $Jac(h(\mathcal{F}))$  and  $Jac(\mathcal{F})$  can be scaled so that the rational unit is a generator in the both modules. Thus,  $Jac(h(\mathcal{F})) \subseteq Jac(\mathcal{F})$  is an inclusion of the  $\mathbb{Z}$ -modules.

(ii) Let us clarify the action,  $\tau_n$ , of the Hecke operators  $T_n \in \mathbb{T}_{\mathbb{Z}}$  on the  $Jac(h(\mathcal{F}))$ . To find a proper formula for such an action, let  $Jac(\mathcal{F}) = \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_g$  and  $Jac(h(\mathcal{F})) = \mathbb{Z}\lambda'_1 + \mathbb{Z}\lambda'_2$ . Then the known action of  $T_n$  on the  $\mathbb{Z}$ -module  $Jac(\mathcal{F})$  extends to the  $\mathbb{Z}$ -module  $Jac(h(\mathcal{F}))$  via the commutative diagram:

$$\begin{array}{ccc} Jac(\mathcal{F}) & \xrightarrow{T_n} & Jac(\mathcal{F}) \\ \downarrow h_* & & \downarrow h_* \\ Jac(h(\mathcal{F})) & \xrightarrow{\tau_n} & Jac(h(\mathcal{F})) \end{array}$$

where  $h_*$  is a projection on the first two coordinates of the basis of the module  $Jac(\mathcal{F})$ . It is known that  $T_n$  acts on the vector  $\lambda = (\lambda_1, \dots, \lambda_g)$  by a matrix  $T_n \in M_g(\mathbb{Z})$ . Since the Hecke operator is a self-adjoint operator, the matrix is a symmetric matrix:

$$\begin{pmatrix} \lambda'_1 \\ \lambda'_2 \\ \vdots \\ \lambda'_g \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1g} \\ t_{12} & t_{22} & \dots & t_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ t_{1g} & t_{2g} & \dots & t_{gg} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_g \end{pmatrix}$$

where  $t_{ij} \in \mathbb{Z}$  are the elements of the matrix  $T_n$ . Thus, one gets  $h_*(\lambda_1, \dots, \lambda_g) = (\lambda_1, \lambda_2, 0, \dots, 0)$  and  $h_*(\lambda'_1, \dots, \lambda'_g) = (\lambda'_1, \lambda'_2, 0, \dots, 0)$ . It is immediate that  $\tau_n = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} \in M_2(\mathbb{Z})$ , i.e. the Hecke operators act on the basis of the module  $Jac(h(\mathcal{F}))$  by the two-by-two symmetric integral matrices.

(iii) Let us show that  $\theta$  is a quadratic irrationality. Indeed, each  $\tau_n$  defines an endomorphism of the  $\mathbb{Z}$ -module  $Jac(h(\mathcal{F}))$ , which is compatible with the multiplication by a real number  $k$ :  $k\lambda_1 = t_{11}\lambda_1 + t_{12}\lambda_2$ ,  $k\lambda_2 = t_{12}\lambda_1 + t_{22}\lambda_2$ . Note that  $\theta = \lambda_2/\lambda_1$ . From the above,  $\theta$  must satisfy the equation  $\theta = \frac{t_{12} + t_{22}\theta}{t_{11} + t_{12}\theta}$ , which is equivalent to a quadratic equation  $t_{12}\theta^2 + (t_{11} - t_{22})\theta - t_{12} = 0$ . The determinant of the latter is  $D = (t_{11} - t_{22})^2 + 4t_{12}^2 \geq 0$ . Thus, the quadratic equation has real roots. These roots cannot be rational, since  $\lambda_1, \lambda_2$  are linearly independent over  $\mathbb{Q}$ . Therefore,  $\theta$  is a quadratic irrationality.  $\square$



## 2 Proof of theorem 1

Let  $R = \text{End}(\mathfrak{m}_H)$  be an order in the real quadratic field  $k = R \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $Cl(R)$  the ideal class group of the order  $R$ . The proof of theorem 1 requires to establish an isomorphism  $Cl(R) \cong \text{Gal}(K|k)$ , where  $\text{Gal}(K|k)$  is the Galois group of the extension  $K|k$ . Note, that the isomorphism will be defined, if one indicates the action of the group  $Cl(R)$  on the generators of the extension  $K|k$ . To see why it is possible, one looks at the basis in  $S_2(\Gamma_0(N))$  consisting of the Anosov-Hecke eigenforms  $\{f_1, \dots, f_g\}$ , rather than at the individual form. Each  $f_i$  produces a Hecke pseudo-lattice and so one gets  $\{\mathfrak{m}_H^{(1)}, \dots, \mathfrak{m}_H^{(g)}\}$  pairwise non-isomorphic Hecke pseudo-lattices, such that  $\text{End}(\mathfrak{m}_H^{(i)}) \cong R$  for all  $i = 1, \dots, g$ . (In other words,  $h_R = g$ , where  $h_R$  is the class number of the order  $R$ .) On the other hand, each Anosov-Hecke eigenform  $f_i$  comes with a Hecke unit,  $\lambda_i \in K$ , so that the collection  $\lambda_1, \dots, \lambda_g$  are the conjugate algebraic integers. The formula  $j(\mathfrak{m}_H^{(i)}) = \lambda_i$  gives the required action of the group  $Cl(R)$  on the generators of the field  $K$ . Let us pass to a detailed argument.

(i) Let  $\lambda_A$  be the Hecke unit associated to the Anosov-Hecke eigenform  $f$ . The characteristic polynomial,  $P(A) \in \mathbb{Z}[x]$ , of the matrix  $A$  is a monic irreducible polynomial with the real roots:

$$P(A) = (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_g), \quad \lambda_i \in K_f. \quad (1)$$

The eigenvectors of the matrix  $A$  correspond to a basis in  $S_2(\Gamma_0(N))$  consisting of the Anosov-Hecke eigenforms  $\{f_1, \dots, f_g\}$ . (We shall always assume, that  $\lambda_1 = \lambda_A$  is the Perron-Frobenius eigenvalue and  $f_1 = f$ .) Each  $f_i$  gives rise to a Hecke pseudo-lattice  $\mathfrak{m}_H^{(i)}$ . We shall define the  $j$ -invariant of  $\mathfrak{m}_H^{(i)}$  as:

$$j(\mathfrak{m}_H^{(i)}) := \lambda_i. \quad (2)$$

(ii) Let  $R = \text{End}(\mathfrak{m}_H)$  and let  $h_R$  be the class number of the order  $R$ . Then  $h_R = g$ , where  $g$  is the genus of the surface  $X_0(N)$ . Indeed, let us show that: (a)  $g \leq h_R$ . The basis  $\{f_1, \dots, f_g\}$  of the Hecke eigenforms gives the Hecke pseudo-lattices  $\mathfrak{m}_H^{(1)}, \dots, \mathfrak{m}_H^{(g)}$ , such that  $\text{End}(\mathfrak{m}_H^{(i)}) = R$ . Thus, the order  $R$  has at least  $g$  ideal classes, i.e.  $g \leq h_R$ . (b) Let us show that  $g \geq h_R$ . Let  $\mathfrak{m}_H^{(1)}, \dots, \mathfrak{m}_H^{(h_R)}$  be a full list of the Hecke pseudo-lattices in the order  $R$ . Since  $R \cong \mathbb{T}_{\mathbb{Z}}/\mathbb{I}$ , where  $\mathbb{I}$  is a fixed ideal in the Hecke ring  $\mathbb{T}_{\mathbb{Z}}$ , we conclude that there exists at least  $f_1, \dots, f_{h_R}$  Anosov-Hecke eigenforms in the space  $S_2(\Gamma_0(N))$ . Thus  $g \geq h_R$ . From (a) and (b), it follows that  $g = h_R$ .

(iii) To finish our proof, let us establish an explicit formula for the isomorphism  $Cl(R) \rightarrow \text{Gal}(K|k)$ . Since the Galois group is an automorphism group of the field  $K$ , it will be enough to find the action of an element  $a \in Cl(R)$  on the generators of  $K$ . Let  $\mathfrak{m}_H^{(i)} \subseteq R$  be a Hecke pseudo-lattice in the order  $R$  and let  $[\mathfrak{m}_H^{(i)}]$  be the ideal class of  $\mathfrak{m}_H^{(i)}$  in  $R$ . Since  $[\mathfrak{m}_H^{(i)}] \in Cl(R)$ , the element  $a * [\mathfrak{m}_H^{(i)}] \in Cl(R)$  for all  $a \in Cl(R)$ . We let  $\mathfrak{m}_H^{(j)}$  be a Hecke pseudo-lattice, such

that  $[\mathfrak{m}_H^{(j)}] = a * [\mathfrak{m}_H^{(i)}]$ . For the sake of brevity, we simply write  $\mathfrak{m}_H^{(j)} = a * [\mathfrak{m}_H^{(i)}]$ . The action of an element  $a \in Cl(R)$  on the generators  $j(\mathfrak{m}_H^{(i)})$  of the field  $K$  is given by the following formula:

$$a * j(\mathfrak{m}_H^{(i)}) := j(a * [\mathfrak{m}_H^{(i)}]), \quad \forall a \in Cl(R). \quad (3)$$

We leave it to the reader to verify that the last formula gives an isomorphism  $Cl(R) \rightarrow Gal(K|k)$ , which completes the proof of theorem 1.  $\square$

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